Error Equations of Inertial Navigation

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This paper derives basic error equations of inertial navigation which apply to any properly constructed inertial navigator. The equations are deduced from the integral equations of inertial navigation by a vectorial analysis. A major result of this analysis is a set of fundamental error propagation equations that has apparently been missed. These equations regard the absolute navigational errors. The conventional velocity and position errors are shown to be transfer errors.

Introduction

THE error propagation equations of inertial navigation are typically derived in relation to a specific frame of reference, most often a locally level frame.[1,2] They are derived here without such confinement. Classical vectorial methods[3] are used to treat the vector errors directly. General error equations applicable to all inertial navigators are obtained, including a new set of absolute error equations.

The discussion begins by reviewing the constructions of inertial navigation. The absolute errors carried by these constructions are identified and are traced through the navigational processes to obtain their equations of propagation. The manner in which an Earth-bound measurement of position and its derivatives can relate to these errors is then examined. This exercise yields a second set of error quantities, termed "transfer errors," which are equally as descriptive of the navigational error state as the absolute errors. The equations propagating the transfer errors are also derived. These equations generalize prior art. Both sets of error equations are subsequently reduced to the "Schuler-tuned" case in which the vertical channel of the navigator is independently constrained. This is followed by a note on the practice of navigational updates and by remarks comparing the different error equations, including those of the prior art.

Navigational Constructions

The basic equation of inertial navigation is

$$\boldsymbol{R}=\int_S\int_S(\boldsymbol{A}-\boldsymbol{G})\mathrm{d}t^2 \quad \text{or} \quad \frac{\mathrm{d}_S{}^2\boldsymbol{R}}{\mathrm{d}t^2}=\boldsymbol{A}-\boldsymbol{G} \tag{1}$$

$\boldsymbol{A}$ represents the sensed acceleration at a point defined by a cluster of accelerometers; $\boldsymbol{R}$ represents the position of this point from Earth center; $\boldsymbol{G}$ represents the specific force of reaction to gravitation at this point; and S represents a space-fixed reference frame.* The subscript on the integral sign identifies the reference frame in which the integration proceeds, i.e., the reference frame to which the components of the integrand are produced for integration, either functionally or actually. Similarly, the subscript on the differential operator d identifies the reference frame in which a vector is noticed for its change of components. These notations are generally applied, i.e., the operations of rate of change and integration over time relative to a reference frame Q are noted as $\mathrm{d}_Q/\mathrm{d}t$ and $\int_Q\mathrm{d}t$.

It is frequent practice to structure the navigational computation so that it gives its output velocity as velocity relative to the Earth. In this scheme

$$\frac{\mathrm{d}_S\boldsymbol{R}}{\mathrm{d}t}=\frac{\mathrm{d}_E\boldsymbol{R}}{\mathrm{d}t}+\omega_E\times\boldsymbol{R}=\boldsymbol{V}_E+\omega_E\times\boldsymbol{R} \tag{2}$$

where E represents an Earth-fixed reference frame, ω_E represents the angular velocity of the Earth, and $\boldsymbol{V}_E=\mathrm{d}_E\boldsymbol{R}/\mathrm{d}t$ represents the velocity relative to the Earth. $\mathrm{d}_S{}^2\boldsymbol{R}/\mathrm{d}t^2$ is expressed from Eq. (2) by

1) Differentiating $\boldsymbol{V}_E$ relative to space in a reference frame F, as

$$\frac{\mathrm{d}_S\boldsymbol{V}_E}{\mathrm{d}t}=\frac{\mathrm{d}_F\boldsymbol{V}_E}{\mathrm{d}t}+\omega_F\times\boldsymbol{V}_E \tag{3}$$

where ω_F represents the angular velocity of F in space.

2) Differentiating $\omega_E\times\boldsymbol{R}$ as

$$\frac{\mathrm{d}_S}{\mathrm{d}t}(\omega_E\times\boldsymbol{R})=\omega_E\times\boldsymbol{V}_E+\omega_E\times(\omega_E\times\boldsymbol{R}) \tag{4}$$

The sum of these results equals $\boldsymbol{A}-\boldsymbol{G}$, so that

$$\frac{\mathrm{d}_F\boldsymbol{V}_E}{\mathrm{d}t}=\boldsymbol{A}-\boldsymbol{G}-(\omega_F+\omega_E)\times\boldsymbol{V}_E-\omega_E\times(\omega_E\times\boldsymbol{R}) \tag{5}$$

Also

$$\frac{\mathrm{d}_S\boldsymbol{R}}{\mathrm{d}t}=\frac{\mathrm{d}_F\boldsymbol{R}}{\mathrm{d}t}+\omega_F\times\boldsymbol{R}=\boldsymbol{V}_E+\omega_E\times\boldsymbol{R} \tag{6}$$

whereby

$$\frac{\mathrm{d}_F\boldsymbol{R}}{\mathrm{d}t}=\boldsymbol{V}_E-(\omega_F-\omega_E)\times\boldsymbol{R} \tag{7}$$

$\boldsymbol{V}_E$ and $\boldsymbol{R}$ are then produced by integrations reciprocal to their expressed rates of change as

$$\boldsymbol{V}_E=\int_F[\boldsymbol{A}-\boldsymbol{G}-(\omega_F+\omega_E)\times\boldsymbol{V}_E-\omega_E\times(\omega_E\times\boldsymbol{R})]\,\mathrm{d}t \tag{8}$$

$$\boldsymbol{R}=\int_F[\boldsymbol{V}_E-(\omega_F-\omega_E)\times\boldsymbol{R}]\,\mathrm{d}t \tag{9}$$

These computations are complemented by an E frame (or S frame) that maintains alignment with the Earth. The E frame sustains the vector ω_E and forms the means of expressing $\boldsymbol{R}$ in Earth coordinates. E also enters into the statement of $\boldsymbol{G}$. The

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*The *space-fixed reference frame* or simply *space* is represented in a rigid body whose angular velocity is measured by gyroscopes as zero.

reference frame F, which is unqualified in Eqs. (8) and (9), may be made the E frame. However, in present practice, E and F are more often distinct frames.

The reference frame(s) of the navigator, whether one or more of E, F, and S, are all based on a body B possessing accelerometers and gyroscopes. It is on this common base that the acceleration A is measured, the reference frames are produced, and communication between them is established.

The E frame is produced as an orthogonal triad of unit vectors e_i, $i=1,2,3$, that satisfy the condition

$$\frac{d_E e_i}{dt}=0 \tag{10}$$

The vectors are constructed in reference to the body B under the relations

$$\frac{d_S e_i}{dt}=\frac{d_E e_i}{dt}+\omega_E\times e_i=\frac{d_B e_i}{dt}+\omega\times e_i \tag{11}$$

ω represents the measured angular velocity of B relative to space. The condition (10) is maintained by

$$\frac{d_B e_i}{dt}=(\omega_E-\omega)\times e_i \tag{12}$$

The e_i are produced accordingly as

$$e_i=e_{io}+\int_B^{t_o,t}(\omega_E-\omega)\times e_i\,dt;\qquad i=1,2,3 \tag{13}$$

Scalar multiplications by the axes b_1, b_2, b_3 of B reduce this equation to its components in B as

$$e_i\cdot b_j=e_{io}\cdot b_j+\int_{t_o}^{t}b_j\cdot(\omega_E-\omega)\times e_i\,dt;\quad i,j=1,2,3 \tag{14}$$

The b_j pass under the integral sign as constants in B.

Equation (13) represents, at least functionally, all the E frame constructions that are maintained gyroscopically.† Within these constructions, the body B may be strapdown or free, space-fixed or Earth-fixed, or locally level. When B is Earth-fixed, itself as the frame E, $\omega=\omega_E$ and the initial values e_{io} [Eq. (13)], which are constant in B, represent the axes of B.

In general, each vector of the triad e_i is expressed by its direction cosines in the reference frame of the body, giving nine direction cosines in all [Eq. (14)]. The direction cosine set is at once the means of defining the frame of the triad with respect to the body and the means of performing a coordinate transformation therebetween. In this circumstance, vectors are transformed between the body frame and the frame of the triad with absolutely no error.

Vector transformations between the body and the F frame are likewise without error. The F frame is produced as

$$f_i=f_{io}+\int_B^{t_o,t}(\omega_F-\omega)\times f_i\,dt;\quad i=1,2,3 \tag{15}$$

so that nine direction cosines in B define the F frame. The set of direction cosines becomes an identity set and $\omega_F=\omega$ when $F=B$.

The F frame is usually made space-fixed, Earth-fixed, or locally level. It is then unlike the E frame only in the locally level case.

†Constructions of the E frame that encounter singularities at the poles are excluded.

The locally level orientation of F is usually established under Eq. (9). This equation is

$$R=\int_F(V-\omega_F\times R)\,dt=\int_S V\,dt \tag{16}$$

where

$$V=V_E+\omega_E\times R \tag{17}$$

Scalar multiplications of the integral in F by the axes f_1, f_2, f_3 of F give the component equations

$$R\cdot f_i=\int(V\cdot f_i-f_i\cdot\omega_F\times R)\,dt;\qquad i=1,2,3 \tag{18}$$

With the notation that $R\cdot f_i=R_i$, $V\cdot f_i=V_i$, and $\omega_F\cdot f_i=\omega_{Fi}$, the equations are

$$R_1=\int(V_1-\omega_{F2}R_3+\omega_{F3}R_2)\,dt$$

$$R_2=\int(V_2-\omega_{F3}R_1+\omega_{F1}R_3)\,dt$$

$$R_3=\int(V_3-\omega_{F1}R_2+\omega_{F2}R_1)\,dt \tag{19}$$

The vector integration accepts any and all ways of satisfying these equations. The locally level constraint is asserted for a spherical Earth model by putting

$$R_1=R_2=0 \tag{20}$$

Then

$$R_3=\int V_3 dt \tag{21}$$

ω_{F1} and ω_{F2} are determined as

$$\omega_{F1}=-V_2/R_3\qquad \omega_{F2}=V_1/R_3 \tag{22}$$

and the integral result is

$$R_3 f_3=R \tag{23}$$

In practice, the structure of R is specified from an ellipsoidal model of the Earth so that f_3 is maintained as the plumb-bob vertical rather than the radial direction from Earth center. Either way, the process remains as

$$R=\int_S V\,dt \tag{24}$$

The locally level alignment of F is merely an output format.

The body frame of the inertial navigator is generally defined by its accelerometers. One accelerometer axis is specified as a body axis. This axis also serves in conjunction with the axis of a second accelerometer to define a body plane. The perpendicular to the first axis in the body plane and the normal to the plane define the second and third body axes.

Cross compensations between the accelerometer outputs adjust the acceleration vector to the body axes. Misalignments between the gyro axes and the body axes are similarly compensated. Residual errors in these calibrations and in others for bias, scale factor, compliance effects, etc. are accounted measurement errors.

Error Propagation Equations

It is apparent that the various constructions of inertial navigation are fundamentally alike, analytically and functionally. The navigational errors by any of them can be considered in common.

The Absolute Error Model

It is necessary to hypothesize a perfect inertial navigator as a base of error comparison. This navigator is denoted by

$$R'=\int_{S'}\int_{S'}(A'-G')\mathrm{d}t^2 \text{ or } \ddot{R}'=A'-G' \tag{25}$$

The primes serve to distinguish the quantities as exact. The overhead dot ($\dot{}$) represents the operation of rate of change relative to S', as $\dot{X}=\mathrm{d}_{S'}X/\mathrm{d}t$. Otherwise, Eq. (25) is just a restatement of Eq. (1). The inertial navigators embody Eq. (25) within limits of error as

$$R=\int_S V\,\mathrm{d}t;\quad V=\int_S (A-G)\,\mathrm{d}t \tag{26}$$

where

$$A=A'+\Delta A \tag{27a}$$

$$G=G'+\Delta G \tag{27b}$$

$$V=\dot{R}'+\Delta V \tag{27c}$$

$$R=R'+\Delta R \tag{27d}$$

and S deviates from S' in response to a body rate measurement

$$\omega=\omega'+\Delta\omega \tag{28}$$

The deviation of S from S' is given by

$$\Delta\theta=\Delta\theta_0-\int_{S'}^{t_o,t}\Delta\omega\,\mathrm{d}t \tag{29}$$

This is immediately evident if the body B is stabilized as

$$\omega=0=\omega'+\Delta\omega \tag{30}$$

Then $B=S$ and B drifts in S' at the rate

$$\omega'=-\Delta\omega \tag{31}$$

When B is space-stabilized, it is invariably initialized to the Earth and, with the aid of a clock, it services the function of the E frame. When B is otherwise, the E frame is constructed.

E is required to conform to a selected orientation E' fixed to the Earth. The process [Eq. (13)] must know the Earth's angular velocity. This information is available as the components of ω_E' in E'.‡ But E' doesn't exist at the navigator. The components of ω_E' as said for E' are necessarily given to E. If E is misaligned from E', the Earth rate vector is misstated as

$$\omega_E=\omega_E'+\Delta\theta\times\omega_E' \tag{32}$$

The axes of E are generated accordingly as

$$\frac{\mathrm{d}_B e_i}{\mathrm{d}t}=(\omega_E'+\Delta\theta\times\omega_E'-\omega'-\Delta\omega)\times e_i \tag{33}$$

‡ω_E' represents the true angular velocity of E', not that of E. It is used instead of $\omega_{E'}'$ as a notational convenience.

Since the angular velocity of B relative to E' is truly $\omega'-\omega_E'$,

$$\frac{\mathrm{d}_B e_i}{\mathrm{d}t}+(\omega'-\omega_E')\times e_i=\frac{\mathrm{d}_{E'}e_i}{\mathrm{d}t}=(-\Delta\omega-\omega_E'\times\Delta\theta)\times e_i \tag{34}$$

This relation says that the angular velocity of E relative to E' is $-\Delta\omega-\omega_E'\times\Delta\theta$. At the small misalignment $\Delta\theta$ between E and E', the relation can be said as

$$\frac{\mathrm{d}_{E'}\Delta\theta}{\mathrm{d}t}=-\Delta\omega-\omega_E'\times\Delta\theta \tag{35}$$

It has the result

$$\frac{\mathrm{d}_{E'}\Delta\theta}{\mathrm{d}t}+\omega_E'\times\Delta\theta=\frac{\mathrm{d}_{S'}\Delta\theta}{\mathrm{d}t}=-\Delta\omega \tag{36}$$

The result agrees with Eq. (29). Equation (36) will be said to model the misalignment of the E frame with the tacit understanding that it applies as well to the S frame.

The processes

$$\frac{\mathrm{d}_S R}{\mathrm{d}t}=V;\qquad \frac{\mathrm{d}_S V}{\mathrm{d}t}=A-G \tag{37}$$

are directly responsible for the velocity and position errors defined by Eqs. (25-27).

As shown in the section on navigational constructions, the velocity and position computations are generally referred to a frame F through the operational substitution

$$\frac{\mathrm{d}_S}{\mathrm{d}t}=\frac{\mathrm{d}_F}{\mathrm{d}t}+\omega_F\times \tag{38}$$

By Eqs. (15) and (28), the frame F is generated in reference to the body B as

$$\frac{\mathrm{d}_B f_i}{\mathrm{d}t}=(\omega_F-\omega'-\Delta\omega)\times f_i;\qquad i=1,2,3 \tag{39}$$

Since ω' represents the true angular velocity of B,

$$\frac{\mathrm{d}_B f_i}{\mathrm{d}t}+\omega'\times f_i=\frac{\mathrm{d}_{S'}f_i}{\mathrm{d}t}=(\omega_F-\Delta\omega)\times f_i \tag{40}$$

As vectors of unit magnitude, the f_i change in S' as

$$\frac{\mathrm{d}_{S'}f_i}{\mathrm{d}t}=\omega_F'\times f_i \tag{41}$$

Therefore,

$$\omega_F=\omega_F'+\Delta\omega \tag{42}$$

and the operational relationship (38) functions as

$$\frac{\mathrm{d}_S}{\mathrm{d}t}=\frac{\mathrm{d}_{S'}}{\mathrm{d}t}+\Delta\omega\times \tag{43}$$

The errors developed in R and V are traced as follows:

$$V=\frac{\mathrm{d}_S R}{\mathrm{d}t}=\left(\frac{\mathrm{d}_{S'}}{\mathrm{d}t}+\Delta\omega\times\right)(R'+\Delta R)$$

$$=\dot{R}'+\Delta\omega\times R'+\Delta\dot{R}+\Delta\omega\times\Delta R \tag{44}$$

$\Delta\omega\times\Delta R$ is dropped as second order, so that

$$V=\dot{R}'+\Delta\omega\times R'+\Delta\dot{R} \tag{45}$$

This relation and the definition that $V=\dot{R}'+\Delta V$ [Eq. (27c)] describe the development of ΔR as

$$\Delta\dot{R}=\Delta V-\Delta\omega\times R' \tag{46}$$

The process generating V is

$$\frac{d_S V}{dt}=\left(\frac{d_{S'}}{dt}+\Delta\omega\times\right)(\dot{R}'+\Delta V)=A-G \tag{47}$$

whereby

$$\ddot{R}'+\Delta\dot{V}+\Delta\omega\times\dot{R}'+\Delta\omega\times\Delta V=A'+\Delta A-G'-\Delta G \tag{48}$$

With the deletion of the second-order term and the removal of the equality $\ddot{R}'=A'-G'$, Eq. (48) becomes

$$\Delta\dot{V}=\Delta A-\Delta G-\Delta\omega\times\dot{R}' \tag{49}$$

ΔG can be said as

$$\Delta G=\Delta R\cdot\text{grad } G' \tag{50}$$

It is negligibly affected by the small dependence of G on the E frame.§ The equation describing the development of ΔV is

$$\Delta\dot{V}=\Delta A-\Delta R\cdot\text{grad } G'-\Delta\omega\times\dot{R}' \tag{51}$$

Equations (36), (46), and (51) constitute the absolute error propagation model of inertial navigation. The model adapts to computation in the F frame as

$$\frac{d_F\Delta\theta}{dt}=-\Delta\omega-\omega_F\times\Delta\theta \tag{52a}$$

$$\frac{d_F\Delta V}{dt}=\Delta A-\Delta R\cdot\text{grad } G-\Delta\omega\times V-\omega_F\times\Delta V \tag{52b}$$

$$\frac{d_F\Delta R}{dt}=\Delta V-\Delta\omega\times R-\omega_F\times\Delta R \tag{52c}$$

The dropping of the primes so that known quantities are represented does not diminish the first-order validity of the equations. The quantities ΔA and $\Delta\omega$ generally receive additional modeling to represent the specific accelerometer and gyro errors that require calibration.

The Transfer Errors

The navigational errors of position and velocity may be described in terms other than their absolute errors. This alternative is found from the applied relationships between an independent measurement of R' and the errors ΔR, ΔV, and $\Delta\theta$.

Independent measurements of R' are generally obtained in terms of Earth coordinates in E'. This includes the cases of checkpoint fixes and fixes by satellite (GPS). When such measures are given to an inertial navigator, they are necessarily referred to the reference frame E of the inertial navigator and are misaligned accordingly. Thus, a measure of R' expressed in Earth coordinates is received at the navigator as

$$R'_m=R'+\Delta\theta\times R' \tag{53}$$

(Note: error of measurement is not considered here.) The position error thereby observed is

$$R-R'_m\equiv\Delta P=R-(R'+\Delta\theta\times R') \tag{54a}$$

$$=\Delta R-\Delta\theta\times R' \tag{54b}$$

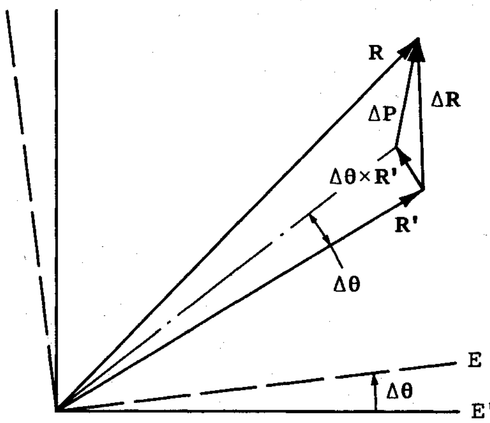

Fig. 1 Navigational errors.

Figure 1 illustrates the misalignment of the Earth reference frame and the position errors that are represented in Eq. (54).

$\Delta\dot{P}$ also represents a comparison like that of ΔP, but only because of Eqs. (36) and (46):

$$\Delta\dot{P}=\Delta\dot{R}+\Delta\omega\times R'-\Delta\theta\times\dot{R}' \tag{55a}$$

$$=\Delta V-\Delta\theta\times\dot{R}' \tag{55b}$$

$$=V-(\dot{R}'+\Delta\theta\times\dot{R}') \tag{55c}$$

$\Delta\dot{P}$ notes the difference between V and a measure of $\dot{R}'$ that is misaligned through the angle $\Delta\theta$.

The same manner of comparison is repeated in

$$\frac{d_{E'}\Delta P}{dt}=V_E-(V'_E+\Delta\theta\times V'_E) \tag{56}$$

This identification follows from the relation

$$\frac{d_{E'}\Delta P}{dt}=\Delta\dot{P}-\omega'_E\times\Delta P \tag{57}$$

From the expression of ΔP [Eq. (54a)],

$$\omega'_E\times\Delta P=\omega'_E\times(R-R'-\Delta\theta\times R')$$

$$=\omega'_E\times R-\omega'_E\times R'-\omega'_E\times(\Delta\theta\times R') \tag{58}$$

By Eq. (32), $\omega'_E=\omega_E-\Delta\theta\times\omega'_E$, whereby

$$\omega'_E\times R=\omega_E\times R-(\Delta\theta\times\omega'_E)\times(R'+\Delta R)$$

$$=\omega_E\times R-(\Delta\theta\times\omega'_E)\times R' \tag{59}$$

With this substitution,

$$\omega'_E\times\Delta P=\omega_E\times R-\omega'_E\times R'-\omega'_E\times(\Delta\theta\times R')$$

$$-(\Delta\theta\times\omega'_E)\times R'$$

$$=\omega_E\times R-\omega'_E\times R'-\Delta\theta\times(\omega'_E\times R') \tag{60}$$

§As stated for a spheroidal model of the Earth, $G=G(R,z)$ where $z=\omega_E/\omega_E$. Therefore,

$$\Delta G=\Delta R\cdot\frac{\partial G}{\partial R}+\Delta z\cdot\frac{\partial G}{\partial z}$$

where

$$\frac{\partial G}{\partial R}\equiv\text{grad } G \quad\text{and}\quad \Delta z=\Delta\theta\times z$$

This relation and the expression of $\Delta\dot{P}$ [Eq. (55c)] bring relation (57) to the result

$$\frac{\mathrm{d}_{E'}\Delta P}{\mathrm{d}t}=(V-\omega_E\times R)-(\dot{R}'-\omega_E'\times R')-\Delta\theta\times(\dot{R}'-\omega_E'\times R')$$

$$=V_E-(V_E'+\Delta\theta\times V_E')$$

$$=\Delta V_E-\Delta\theta\times V_E' \tag{61}$$

The above statements of $\Delta\dot{P}$ and $\mathrm{d}_{E'}\Delta P/\mathrm{d}t$ [Eqs. (55) and (61)] are analytical consequences of the definition of ΔP at Eq. (54). It is evident that ΔP, $\Delta\dot{P}$, $\mathrm{d}_{E'}\Delta P/\mathrm{d}t$ and ΔR, ΔV, ΔV_E, together with $\Delta\theta$, are equally descriptive error sets. They go from one to the other through the relations

$$\Delta P+\Delta\theta\times R'=\Delta R \tag{62a}$$

$$\Delta\dot{P}+\Delta\theta\times\dot{R}'=\Delta V \tag{62b}$$

$$\frac{\mathrm{d}_{E'}\Delta P}{\mathrm{d}t}+\Delta\theta\times V_E'=\Delta V_E \tag{62c}$$

The quantities ΔP, $\Delta\dot{P}$, and $\mathrm{d}_{E'}\Delta P/\mathrm{d}t$ will be referred to as "transfer errors." They represent the navigational errors that are observed if measurements of R', $\dot{R}'$, and V_E' in E' are given to E by a simple *transfer* of coordinates.

The Transfer Error Model

The transfer errors are propagated by the inertial errors represented in

$$\Delta\ddot{P}=\Delta\dot{V}-\frac{\mathrm{d}_{S'}}{\mathrm{d}t}(\Delta\theta\times\dot{R}')$$

$$=\Delta\dot{V}+\Delta\omega\times\dot{R}'-\Delta\theta\times\ddot{R}' \tag{63}$$

When the errors causing $\Delta\dot{V}$ are substituted from Eq. (51),

$$\Delta\ddot{P}=\Delta A-\Delta R\cdot\mathrm{grad}\ G'-\Delta\theta\times\ddot{R}' \tag{64}$$

The inverse square law

$$G=(\mu/R^3)R \tag{65}$$

represents the Earth's gravitation sufficiently well to model the error $\Delta R\cdot\mathrm{grad}\ G'$. Therefore,

$$\mathrm{grad}\ G'=\frac{\mu}{R'^3}\frac{\partial R}{\partial R}-\frac{3\mu}{R'^4}\frac{\partial R}{\partial R}R'$$

$$=\frac{\mu}{R'^3}U-\frac{3\mu}{R'^5}R'R' \tag{66}$$

where

$$\frac{\partial R}{\partial R}=U=\text{unity tensor};\quad \frac{\partial R}{\partial R}=\frac{R'}{R'}$$

and

$$\Delta R\cdot\mathrm{grad}\ G'=(\Delta P+\Delta\theta\times R')\cdot\mathrm{grad}\ G'$$

$$=\Delta P\cdot\mathrm{grad}\ G'+\Delta\theta\times G' \tag{67}$$

Eq. (67) brings Eq. (64) to the result

$$\Delta\ddot{P}=\Delta A-\Delta\theta\times(\ddot{R}'+G')-\Delta P\cdot\mathrm{grad}\ G'$$

$$=\Delta A-\Delta\theta\times A'-\Delta P\cdot\mathrm{grad}\ G' \tag{68}$$

This result is the basic transfer error equation for inertial navigators.

The equation may be reformed to suit the construction of the inertial navigator. In connection with navigation in a frame F having the velocity output V_E, $\Delta\ddot{P}$ may be said as

$$\Delta\ddot{P}=\left(\frac{\mathrm{d}_F}{\mathrm{d}t}+\omega_F\times\right)\frac{\mathrm{d}_E\Delta P}{\mathrm{d}t}+\left(\frac{\mathrm{d}_E}{\mathrm{d}t}+\omega_E\times\right)(\omega_E\times\Delta P)$$

$$=\frac{\mathrm{d}_F}{\mathrm{d}t}\left(\frac{\mathrm{d}_E\Delta P}{\mathrm{d}t}\right)+(\omega_F+\omega_E)\times\frac{\mathrm{d}_E\Delta P}{\mathrm{d}t}+\omega_E\times(\omega_E\times\Delta P) \tag{69}$$

$\mathrm{d}_E\Delta P/\mathrm{d}t$ is then treated in the frame F under the equation

$$\frac{\mathrm{d}_F}{\mathrm{d}t}\left(\frac{\mathrm{d}_E\Delta P}{\mathrm{d}t}\right)=\Delta A-\Delta\theta\times A-\Delta P\cdot\mathrm{grad}\ G$$

$$-(\omega_F+\omega_E)\times\frac{\mathrm{d}_E\Delta P}{\mathrm{d}t}-\omega_E\times(\omega_E\times\Delta P) \tag{70}$$

ΔP and $\Delta\theta$ are treated in F under the equations

$$\frac{\mathrm{d}_F\Delta P}{\mathrm{d}t}=\frac{\mathrm{d}_E\Delta P}{\mathrm{d}t}+(\omega_E-\omega_F)\times\Delta P \tag{71}$$

$$\frac{\mathrm{d}_F\Delta\theta}{\mathrm{d}t}=-\Delta\omega-\omega_F\times\Delta\theta \tag{52a}$$

The primes are dropped since the equations have only first-order error terms.

The "Schuler-Tuned" Case

For "Schuler-tuned" inertial navigators, the differential equations for the velocity and position errors have to be abridged. The Schuler-tuned navigators receive an independent measure of altitude which is made to prevail over the magnitude of the inertially generated position output. Consequently, the vertical position error

$$(R/R)\cdot\Delta P=(R/R)\cdot\Delta R$$

and its derivative must be denied representation in the outputs of the error equations since they are otherwise determined, presumably as zero.

The Schuler-Tuned Transfer Error Model

For convenience, the unit vector R/R is identified as

$$k=R/R \tag{72}$$

$k\cdot\Delta P$ is said to be zero. Since it is a scalar, its rate of change is the same in any frame of reference as

$$\frac{\mathrm{d}_Q}{\mathrm{d}t}(k\cdot\Delta P)=0 \tag{73}$$

Q may be selected as a frame of reference K in which k is constant, e.g., a frame that is aligned to R. Then, k can pass across the differential operator so that

$$\frac{d_Q}{dt}(\boldsymbol{k}\cdot\Delta\boldsymbol{P})=\boldsymbol{k}\cdot\frac{d_K\Delta\boldsymbol{P}}{dt}=0 \tag{74}$$

This statement and the relation

$$\frac{d_K\Delta\boldsymbol{P}}{dt}=\Delta\dot{\boldsymbol{P}}-\boldsymbol{\omega}_K\times\Delta\boldsymbol{P} \tag{75}$$

bring the result

$$\boldsymbol{k}\cdot\Delta\dot{\boldsymbol{P}}=\boldsymbol{k}\cdot\boldsymbol{\omega}_K\times\Delta\boldsymbol{P}=-\boldsymbol{\omega}_K\cdot\boldsymbol{k}\times\Delta\boldsymbol{P} \tag{76}$$

Therefore, $\boldsymbol{k}\cdot\Delta\dot{\boldsymbol{P}}$ is known when $\boldsymbol{k}\times\Delta\boldsymbol{P}$ is known, and since $\boldsymbol{k}\cdot\Delta\boldsymbol{P}=0$ is given, the resident unknowns in $\Delta\boldsymbol{P}$ and $\Delta\dot{\boldsymbol{P}}$ are $\boldsymbol{k}\times\Delta\boldsymbol{P}$ and $\boldsymbol{k}\times\Delta\dot{\boldsymbol{P}}$.

The equations for $\boldsymbol{k}\times\Delta\boldsymbol{P}$ and $\boldsymbol{k}\times\Delta\dot{\boldsymbol{P}}$ are readily obtained in the reference frame K, as follows:

$$\begin{aligned}\frac{d_K}{dt}(\boldsymbol{k}\times\Delta\boldsymbol{P})&=\boldsymbol{k}\times(\Delta\dot{\boldsymbol{P}}-\boldsymbol{\omega}_K\times\Delta\boldsymbol{P})\\&=\boldsymbol{k}\times\Delta\dot{\boldsymbol{P}}-(\boldsymbol{k}\cdot\Delta\boldsymbol{P})\boldsymbol{\omega}_K+(\boldsymbol{\omega}_K\cdot\boldsymbol{k})\Delta\boldsymbol{P}\end{aligned} \tag{77}$$

Since

$$\Delta\boldsymbol{P}=-\boldsymbol{k}\times(\boldsymbol{k}\times\Delta\boldsymbol{P}) \tag{78}$$

Eq. (77) becomes

$$\frac{d_K}{dt}(\boldsymbol{k}\times\Delta\boldsymbol{P})=\boldsymbol{k}\times\Delta\dot{\boldsymbol{P}}-(\boldsymbol{\omega}_K\cdot\boldsymbol{k})\boldsymbol{k}\times(\boldsymbol{k}\times\Delta\boldsymbol{P}) \tag{79}$$

Also,

$$\begin{aligned}\frac{d_K}{dt}(\boldsymbol{k}\times\Delta\dot{\boldsymbol{P}})&=\boldsymbol{k}\times(\Delta\ddot{\boldsymbol{P}}-\boldsymbol{\omega}_K\times\Delta\dot{\boldsymbol{P}})\\&=\boldsymbol{k}\times\Delta\ddot{\boldsymbol{P}}-(\boldsymbol{k}\cdot\Delta\dot{\boldsymbol{P}})\boldsymbol{\omega}_K+(\boldsymbol{\omega}_K\cdot\boldsymbol{k})\Delta\dot{\boldsymbol{P}}\end{aligned} \tag{80}$$

Since

$$\begin{aligned}\Delta\dot{\boldsymbol{P}}&=-\boldsymbol{k}\times(\boldsymbol{k}\times\Delta\dot{\boldsymbol{P}})+(\boldsymbol{k}\cdot\Delta\dot{\boldsymbol{P}})\boldsymbol{k}\\&=-\boldsymbol{k}\times(\boldsymbol{k}\times\Delta\dot{\boldsymbol{P}})-(\boldsymbol{\omega}_K\cdot\boldsymbol{k}\times\Delta\boldsymbol{P})\boldsymbol{k}\end{aligned} \tag{81}$$

Eq. (80) becomes

$$\begin{aligned}\frac{d_K}{dt}(\boldsymbol{k}\times\Delta\dot{\boldsymbol{P}})&=\boldsymbol{k}\times\Delta\ddot{\boldsymbol{P}}+(\boldsymbol{\omega}_K\cdot\boldsymbol{k}\times\Delta\boldsymbol{P})\boldsymbol{\omega}_K\\&\quad-\boldsymbol{\omega}_K\cdot\boldsymbol{k}[\boldsymbol{k}\times(\boldsymbol{k}\times\Delta\dot{\boldsymbol{P}})+(\boldsymbol{\omega}_K\cdot\boldsymbol{k}\times\Delta\boldsymbol{P})\boldsymbol{k}]\\&=\boldsymbol{k}\times\Delta\ddot{\boldsymbol{P}}-(\boldsymbol{\omega}_K\cdot\boldsymbol{k})\boldsymbol{k}\times(\boldsymbol{k}\times\Delta\dot{\boldsymbol{P}})\\&\quad+(\boldsymbol{\omega}_K\cdot\boldsymbol{k}\times\Delta\boldsymbol{P})\boldsymbol{k}\times(\boldsymbol{\omega}_K\times\boldsymbol{k})\end{aligned} \tag{82}$$

The term $\boldsymbol{k}\times\Delta\ddot{\boldsymbol{P}}$ is given by Eqs. (66) and (68) as

$$\boldsymbol{k}\times\Delta\ddot{\boldsymbol{P}}=\boldsymbol{k}\times\Delta\boldsymbol{A}-\boldsymbol{k}\times(\Delta\boldsymbol{\theta}\times\boldsymbol{A})-\frac{\mu}{R^3}\boldsymbol{k}\times\Delta\boldsymbol{P} \tag{83}$$

where

$$\begin{aligned}\boldsymbol{k}\times(\Delta\boldsymbol{P}\cdot\text{grad }\boldsymbol{G})&=\boldsymbol{k}\times\left[\frac{\mu}{R^3}\Delta\boldsymbol{P}-\frac{3\mu}{R^3}(\Delta\boldsymbol{P}\cdot\boldsymbol{k})\boldsymbol{k}\right]\\&=\frac{\mu}{R^3}\boldsymbol{k}\times\Delta\boldsymbol{P}\end{aligned} \tag{84}$$

The end results are

$$\frac{d_K}{dt}(\boldsymbol{k}\times\Delta\boldsymbol{P})=\boldsymbol{k}\times\Delta\dot{\boldsymbol{P}}-(\boldsymbol{\omega}_K\cdot\boldsymbol{k})\boldsymbol{k}\times(\boldsymbol{k}\times\Delta\boldsymbol{P}) \tag{79}$$

and

$$\begin{aligned}\frac{d_K}{dt}(\boldsymbol{k}\times\Delta\dot{\boldsymbol{P}})&=\boldsymbol{k}\times\Delta\boldsymbol{A}-\boldsymbol{k}\times(\Delta\boldsymbol{\theta}\times\boldsymbol{A})-\frac{\mu}{R^3}\boldsymbol{k}\times\Delta\boldsymbol{P}\\&\quad-(\boldsymbol{\omega}_K\cdot\boldsymbol{k})\boldsymbol{k}\times(\boldsymbol{k}\times\Delta\dot{\boldsymbol{P}})+(\boldsymbol{\omega}_K\cdot\boldsymbol{k}\times\Delta\boldsymbol{P})\boldsymbol{k}\times(\boldsymbol{\omega}_K\times\boldsymbol{k})\end{aligned} \tag{85}$$

$\boldsymbol{\omega}_K\cdot\boldsymbol{k}$ represents an imposed value of angular rate, since it is otherwise undefined [cf. Eqs. (16-23)]. $\boldsymbol{\omega}_K\times\boldsymbol{k}$ is usually small enough so that the term

$$(\boldsymbol{\omega}_K\cdot\boldsymbol{k}\times\Delta\boldsymbol{P})\boldsymbol{k}\times(\boldsymbol{\omega}_K\times\boldsymbol{k})$$

in Eq. (85) can be dropped as negligible. The substitution

$$\begin{aligned}\boldsymbol{k}\times\Delta\dot{\boldsymbol{P}}&=\boldsymbol{k}\times\left(\frac{d_E\Delta\boldsymbol{P}}{dt}+\boldsymbol{\omega}_E\times\Delta\boldsymbol{P}\right)\\&=\boldsymbol{k}\times\frac{d_E\Delta\boldsymbol{P}}{dt}+(\boldsymbol{\omega}_E\cdot\boldsymbol{k})\boldsymbol{k}\times(\boldsymbol{k}\times\Delta\boldsymbol{P})\end{aligned} \tag{86}$$

converts Eqs. (79) and (85) to the velocity error $\boldsymbol{k}\times(d_E\Delta\boldsymbol{P}/dt)$ as

$$\frac{d_K}{dt}(\boldsymbol{k}\times\Delta\boldsymbol{P})=\boldsymbol{k}\times\frac{d_E\Delta\boldsymbol{P}}{dt}+(\boldsymbol{\omega}_E\cdot\boldsymbol{k}-\boldsymbol{\omega}_K\cdot\boldsymbol{k})\boldsymbol{k}\times(\boldsymbol{k}\times\Delta\boldsymbol{P}) \tag{87}$$

$$\begin{aligned}\frac{d_K}{dt}\left(\boldsymbol{k}\times\frac{d_E\Delta\boldsymbol{P}}{dt}\right)&=\boldsymbol{k}\times\Delta\boldsymbol{A}-\boldsymbol{k}\times(\Delta\boldsymbol{\theta}\times\boldsymbol{A})-\frac{\mu}{R^3}\boldsymbol{k}\times\Delta\boldsymbol{P}\\&\quad-(\boldsymbol{\omega}_E\cdot\boldsymbol{k}+\boldsymbol{\omega}_K\cdot\boldsymbol{k})\boldsymbol{k}\times\left(\boldsymbol{k}\times\frac{d_E\Delta\boldsymbol{P}}{dt}\right)\end{aligned} \tag{88}$$

The terms of this set that carry a product or square of $\boldsymbol{\omega}_E$ and $\boldsymbol{\omega}_K\times\boldsymbol{k}$ have been dropped as negligible.

The Schuler-Tuned Absolute Error Model

The absolute error model for the Schuler-tuned case is derived similarly as the transfer error model. Since $\boldsymbol{k}\cdot\Delta\dot{\boldsymbol{R}}=\boldsymbol{k}\cdot\Delta\boldsymbol{V}$ [Eqs. (46) and (72)], the derivation is exactly as above up to the results

$$\frac{d_K}{dt}(\boldsymbol{k}\times\Delta\boldsymbol{R})=\boldsymbol{k}\times\Delta\dot{\boldsymbol{R}}-(\boldsymbol{\omega}_K\cdot\boldsymbol{k})\boldsymbol{k}\times(\boldsymbol{k}\times\Delta\boldsymbol{R}) \tag{89}$$

$$\begin{aligned}\frac{d_K}{dt}(\boldsymbol{k}\times\Delta\boldsymbol{V})&=\boldsymbol{k}\times\Delta\dot{\boldsymbol{V}}-(\boldsymbol{\omega}_K\cdot\boldsymbol{k})\boldsymbol{k}\times(\boldsymbol{k}\times\Delta\boldsymbol{V})\\&\quad+(\boldsymbol{\omega}_K\cdot\boldsymbol{k}\times\Delta\boldsymbol{R})\boldsymbol{k}\times(\boldsymbol{\omega}_K\times\boldsymbol{k})\end{aligned} \tag{90}$$

The small term having a double multiplication by $\boldsymbol{\omega}_K$ may be dropped as negligible. From the general absolute error equations [Eqs. (46) and (51)],

$$\boldsymbol{k}\times\Delta\dot{\boldsymbol{R}}=\boldsymbol{k}\times\Delta\boldsymbol{V}-\boldsymbol{k}\times(\Delta\boldsymbol{\omega}\times\boldsymbol{R}) \tag{91}$$

$$\boldsymbol{k}\times\Delta\dot{\boldsymbol{V}}=\boldsymbol{k}\times\Delta\boldsymbol{A}-\frac{\mu}{R^3}\boldsymbol{k}\times\Delta\boldsymbol{R}-\boldsymbol{k}\times(\Delta\boldsymbol{\omega}\times\boldsymbol{V}) \tag{92}$$

which complete the preceding equations as

$$\begin{aligned}\frac{d_K}{dt}(\boldsymbol{k}\times\Delta\boldsymbol{R})&=\boldsymbol{k}\times\Delta\boldsymbol{V}-\boldsymbol{k}\times(\Delta\boldsymbol{\omega}\times\boldsymbol{R})\\&\quad-(\boldsymbol{\omega}_K\cdot\boldsymbol{k})\boldsymbol{k}\times(\boldsymbol{k}\times\Delta\boldsymbol{R})\end{aligned} \tag{93}$$

$$\frac{\mathrm{d}_K}{\mathrm{d}t}(k\times\Delta V)=k\times\Delta A-\frac{\mu}{R^3}k\times\Delta R-k\times(\Delta\omega\times V)$$

$$-(\omega_K\cdot k)k\times(k\times\Delta V) \tag{94}$$

In the absence of ΔA and $\Delta\omega$, $k\times\Delta R$ and $k\times\Delta V$ develop almost independently of the navigational motion as

$$\frac{\mathrm{d}_K}{\mathrm{d}t}(k\times\Delta R)=k\times\Delta V-(\omega_K\cdot k)k\times(k\times\Delta R) \tag{95}$$

$$\frac{\mathrm{d}_K}{\mathrm{d}t}(k\times\Delta V)=-\frac{\mu}{R^3}k\times\Delta R-(\omega_K\cdot k)k\times(k\times\Delta V) \tag{96}$$

This result accords with Schuler's thesis on the behavior of inertial vertical-indicating devices that are appropriately tuned.[4] When

$$\mu/R^3\equiv\omega_s^2 \tag{97}$$

is constant or nearly so, the solution of Eqs. (95) and (96) is

$$k\times\Delta R=\left(k\times\Delta R_0\cos\omega_s t+k\times\Delta V_0\frac{\sin\omega_s t}{\omega_s}\right)\cos\zeta$$

$$-k\times\left(k\times\Delta R_0\cos\omega_s t+k\times\Delta V_0\frac{\sin\omega_s t}{\omega_s}\right)\sin\zeta \tag{98}$$

$$k\times\Delta V=(-k\times\Delta R_0\omega_s\sin\omega_s t+k\times\Delta V_0\cos\omega_s t)\cos\zeta$$

$$+k\times(k\times\Delta R_0\,\omega_s\sin\omega_s t-k\times\Delta V_0\cos\omega_s t)\sin\zeta \tag{99}$$

$k\times\Delta R_0$ and $k\times\Delta V_0$ are constants in K and

$$\zeta=\int_0^t\omega_K\cdot k\,\mathrm{d}t \tag{100}$$

Navigational Updates

The error propagation model is ordinarily combined with independent navigational error measurements and their models to form a Kalman filter. The Kalman filter calculates more or less optimum estimates of the modeled errors. These estimates are used to update the navigator.

The several updates are not necessarily made independently of each other. For example, an estimate of $\Delta\theta$ may be applied so that it turns the frame E toward E', in accordance with Fig. 1. This action may or may not simultaneously turn the vector R. The navigator may operate either way, depending on its construction. If R is turned together with the frame E, the appropriate position update is the estimate of the transfer error ΔP applied negatively. If R does not move with the E frame, its update is properly applied by the estimate of ΔR. The estimates of ΔR and ΔP are interchangeable through the relation $\Delta R=\Delta P+\Delta\theta\times R$ [Eq. (62a)]. The velocity update is similarly dealt with. The general rule is that the superimposed updates to the navigator's position, velocity, and E frame must negate the absolute error estimates of these quantities.

Remarks

The navigational error models that are prominently discussed in the literature are the "psi" model and the "phi" model. They are named for the notation used to identify their angles of misalignment. Both models are associated with locally level frames of reference.

The counterpart of these models in this paper is the transfer error model. But it comes with significantly different interpretations.

ψ is represented in the literature as a misalignment between a "computer frame" and a "platform frame." The platform frame is recognized in the broader concept of this paper's F frame but the computer frame is of no interest whatever. The reference frames considered here are all independently identified frames whereas the computer frame exists as the creature of ψ if it exists at all. However, ψ itself exists and if it is identified as the misalignment of the E frame, i.e. as $\Delta\theta$ (Fig. 1), the ψ model is reconciled with the transfer error model of Eqs. (70), (71), and (52a).

The ϕ model is a restatement of the ψ model with a minor change of variable. In the terms of Fig. 1, ϕ is defined as

$$\phi=\Delta\theta+(R/R^2)\times\Delta P \tag{101}$$

It represents the angle of tilt of R from R' (Fig. 1) plus the vertical component of $\Delta\theta$. When F is specified as a locally level frame and $\Delta\theta$ is replaced by

$$\phi-(R/R^2)\times\Delta P$$

Eqs. (70), (71), and (52a) become the ϕ model.

To the author's knowledge, the absolute error equations (46) and (51) or (52b,c) or any adaptation of them are nowhere represented in the prior literature. These equations appear to be more convenient to use in Kalman filters than the transfer error equations.

The equations that substantially determine the relative convenience between the transfer model and the absolute model are

$$\Delta\ddot{P}=\Delta A-\Delta\theta\times A-\Delta P\cdot\mathrm{grad}\,G \tag{68}$$

$$\Delta\dot{V}=\Delta A-\Delta\omega\times V-\Delta R\cdot\mathrm{grad}\,G \tag{51}$$

The computationally distinctive terms are $\Delta\theta\times A$ and $\Delta\omega\times V$. $\Delta\theta$ gathers the input of $\Delta\omega$ cumulatively and A tends to be more variable than V. Both circumstances suggest that $\Delta\omega\times V$ may be integrated for various models of $\Delta\omega$ more readily than $\Delta\theta\times A$. However, this is not a conclusive judgment for all cases.

No attempt has been made to reform the absolute error model in terms of ΔV_E rather than ΔV. The model of ΔV and ΔR remains perfectly serviceable when the navigational velocity output is V_E. The statement

$$\Delta V_E=(V-\omega_E\times R)-(V'-\omega_E'\times R')$$

$$=\Delta V-(\omega_E\times R-\omega_E'\times R') \tag{102}$$

and the relation

$$\omega_E\times R=(\omega_E'+\Delta\theta\times\omega_E')\times R$$

$$=\omega_E'\times(R'+\Delta R)+(\Delta\theta\times\omega_E')\times R \tag{103}$$

combine to represent ΔV_E as

$$\Delta V_E=\Delta V-\omega_E\times\Delta R-(\Delta\theta\times\omega_E)\times R \tag{104a}$$

$$=\Delta V-U\times\omega_E\cdot\Delta R+[(R\cdot\omega_E)U-\omega_E R]\cdot\Delta\theta \tag{104b}$$

Measurements or updates of ΔV_E may be processed accordingly.

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